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ADDITIVE *K*-COLORABLE EXTENSIONS OF THE RATIONAL PLANE

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Let F be a field, $\mathbb{Q} \subset F \subset \mathbb{R}$ and consider F^d as a graph with vertices the points of F^d and an edge between two points if their Euclidean distance is 1. Let $\Sigma_0(F^d)$ be the subgroup of F^d generated by the unit vectors ξ. If G is a group of order k, then a group homomorphism $v:\Sigma_0(F^d) \to G$ for which $v(\xi) \neq 0$ whenever $\|\xi\| = 1$ is said to be an additive k-coloring of F^d. The known 2 and 4-colorings of $\mathbb{Q}^3$ and $\mathbb{Q}^4$ respectively, are shown to be additive. If N is a square free integer, then it is shown that $\mathbb{Q}(\sqrt{N})^2$ has an additive 2-coloring iff $N \neq 3 \bmod 4$. If $N \neq 2 \bmod 3$, then $\mathbb{Q}(\sqrt{N})^2$ has an additive 3-coloring. Hence, it follows that the chromatic number of $\mathbb{Q}(\sqrt{3})^2$ is 3. The existence of additive colorings on $\mathbb{Q}(\sqrt{N})^2$ for the remaining cases is also discussed.

Additive k-colorings constrain cycles in F^d to satisfy group identities. Hence, it is shown for example, that if F^2 is 2-colorable and if $\sqrt{2} \notin F$, then F^2 contains no regular polygon except for the square. This generalizes the classical result known for the rational plane.

1. Introduction

Let F be a subfield of the reals $\mathbb{R}$ so that $\mathbb{Q} \subset F \subset \mathbb{R}$ where $\mathbb{Q}$ is the field of rationals. By F^d is meant the collection of points in the real Euclidean space $\mathbb{R}^d$ all of whose d coordinates are in F. These points will be interpreted as vertices of a graph, also denoted by F^d, with an edge between the vertices P and Q if their Euclidean distance $\|P - Q\| = 1$. The connected component containing the origin will be denoted by $\Sigma_0(F^d)$. Hence, this is the subgroup in F^d generated by all the unit vectors.

If k is a positive integer, a k-coloring of F^d is an assignment v of the vertices to a set of k elements so that $v(P) \neq v(Q)$ whenever $\|P - Q\| = 1$. It is easy to see that F^d can be k-colored iff $\Sigma_0(F^d)$ can be k-colored. A group homomorphism $v:\Sigma_0(F^d) \to G$ to an additive group G of order k so that $v(\xi) \neq 0$ whenever $\|\xi\| = 1$, is said to be an additive k-coloring of F^d.

An additive k-coloring clearly provides a k-coloring of F^d. The converse of this is not true although it easy to see that F^d can be 2-colored iff it posesses an additive 2-coloring. It is shown, for example, that $\mathbb{Q}^2$, $\mathbb{Q}^3$ and $\mathbb{Q}^4$ have additive 2, 2 and 4-colorings respectively. That these are also the chromatic numbers of the spaces was known to Woodall [9], Johnson [7] and to Benda and Perles [3] respectively. Quite generally it is shown that if α is a real transcendental over F, then F^d has an additive k-coloring iff $F(\alpha)^d$ does. The same statement holds true for non-additive k-colorings. Additive k-colorings place restrictions on the cycles in F^d since these must now satisfy group identities. Thus, for example, if F^2 is two colorable and $\sqrt{2} \notin F$, then F^2 contains no regular polygon except for the square. In view of the fact that $\mathbb{Q}^2$ is two-colorable, this generalizes the known result for the rational plane [6, p. 4].

This paper considers possible additive k-colorings of the plane formed by a real quadratic extension of $\mathbb{Q}$. If N is a square free positive integer, it is shown that the plane $\mathbb{Q}(\sqrt{N})^2$ has an additive 2-coloring iff $N \neq 3 \bmod 4$. That $\mathbb{Q}(\sqrt{N})^2$ has a two coloring in this case also follows by a recent result of Johnson's in [8]. The converse answers a problem posed by him in that paper. If $N \neq 2 \bmod 3$, it is shown here that $\mathbb{Q}(\sqrt{N})^2$ has an additive 3-coloring. It follows, for example, that the chromatic number of $\mathbb{Q}(\sqrt{3})^2$ is 3. Furthermore, if $N = 3 \bmod 8$, then $\mathbb{Q}(\sqrt{N})^2$ has an additive 4-coloring and therefore places 4 as a bound on its chromatic number. In the remaining cases, it is shown that $1/k \in \Sigma_0(\mathbb{Q}(\sqrt{N})^2)$ for $1 \leq k \leq 6$. Hence, no additive k-coloring for these values of k can exist since if $v : \Sigma_0(\mathbb{Q}(\sqrt{N})^2) \to G$ is such a coloring, then $k \cdot v(1/k) = v(1) \neq 0$ and yet $k \cdot G = 0$. In fact, $\mathbb{Q}(\sqrt{167})^2$ does not allow an additive k-coloring for $k < 11$.

In view of the fact that the chromatic number of $\mathbb{R}^2$ is at least 4 but not more than 7 [5, p. 16], the above example among others shows that additive k-colorings even on quadratic extensions of $\mathbb{Q}$ are too special to provide k-colorings in most cases. This still begs the general question, however, whether some algebraic criterion can be found to determine the chromatic number of F^2 where F is a subfield of $\mathbb{R}$, of finite degree over $\mathbb{Q}$. That the answer to this question would suffice to find the chromatic number of $\mathbb{R}^2$ has already been noted by Benda and Perles in [2]. The assertion also follows easily here from the result of de Bruijn and Erdös in [1] which states that the chromatic number of $\mathbb{R}^2$ is the same as the number of colors required to paint any finite number of points in $\mathbb{R}^2$, and the above theorem on adjoining transcendentals.

2. Additive k-colorings

Let $\Sigma_0(F^d)$ be the points P in the graph F^d for which there exists a path in F^d from the origin 0 to P. It should be emphasized that this means there exists a sequence of points $0 = P_0, \ldots, P_n = P$ in F^d so that $\|P_j - P_{j-1}\| = 1$ for $j = 1, \ldots, n$. Clearly,

$$\Sigma_0(F^d) = \left\{ \xi : \xi = \sum_{j=1}^{n} \xi_j,\ \xi_j \in F^d \text{ and } \|\xi_j\| = 1 \text{ for all } j \right\}$$

and it forms an additive subgroup of F^d. The translates $\Sigma_Q(F^d) = Q + \Sigma_0(F^d)$ identify the points P in F^d which may be reached by a path in F^d from the point Q in F^d.

Since it is always assumed that the field $F \subset \mathbb{R}$, the plane F^2 may be identified with the field of complex numbers $F(i)$. Hence, $\Sigma_0(F^2)$ may be identified as sums of complex numbers in $F(i)$ of norm 1 which form a ring containing the integers $\mathbb{Z}$.

In what follows, any group G is considered to be abelian and additive with identity element 0. The ring $\mathbb{Z}/(2)$ will denote the ring of integers modulo 2 and $[n]$ the equivalence class of an integer n modulo 2.

Theorem 1. i. *If F^d has an additive k-coloring, then F^d can be k-colored.*

ii. *F^d can be 2-colored iff there exists an additive map $v : \Sigma_0(F^d) \to \mathbb{Z}/(2)$ so that* $v(\xi) = [1]$ if $\|\xi\| = 1$.

iii. *The plane F^2 can be 2-colored iff there exists a non-zero ring homomorphism $v : \Sigma_0(F^2) \to \mathbb{Z}/(2)$.*

Proof. Since the connected components of F^d are translates of $\Sigma_0(F^d)$, to k-color F^d it suffices to k-color $\Sigma_0(F^d)$. The existence of a group homomorphism $v : \Sigma_0(F^d) \to G$ where G is of order k, assures that if ξ, $\eta \in \Sigma_0(F^d)$ and if $\|\xi - \eta\| = 1$, then $v(\xi - \eta) \neq 0$. Hence, $v(\xi) \neq v(\eta)$ and therefore v assigns a k-coloring to $\Sigma_0(F^d)$. This provides the proof for part (i).

To show (ii), note that if F^d can be 2-colored, then the sequence of points in a path must have alternating colors and hence every cycle must have an even number of edges. It follows that if $\xi = \sum_{j=1}^{n} \xi_j \in \Sigma_0(F^d)$, with $\|\xi_j\| = 1$, the map $v : \Sigma_0(F^d) \to \mathbb{Z}/(2)$ for which $v(\xi) = [n]$ is well defined. It is easily seen to be additive and $v(\xi) = [1]$ if $\|\xi\| = 1$. Using part (i), this proves (ii).

For a 2-colorable plane F^2, the additive map from (ii) is easily seen to be a ring homomorphism since by assumption $v(\xi_1 \cdot \xi_2) = [1]$ if $\|\xi_1\| = \|\xi_2\| = 1$. But conversely, given any non-zero homomorphism on the ring $\Sigma_0(F^2)$, it is clearly additive and if $\|\xi\| = 1$, then ξ has an inverse in $\Sigma_0(F^2)$ and therefore $v(\xi) = 1$. It follows from (i) that F^2 is 2-colorable and proves part (iii). $\square$

Let $\mathbb{Z}_{(2)}$ be the ring of fractions of $\mathbb{Q}$ with odd denominators. This is a local ring with maximal ideal (2). Let $\epsilon : \mathbb{Z}_{(2)} \to \mathbb{Z}/(2)$ be the evaluation at the maximal ideal.

Note that for any integer n, $n = 0 \bmod 2$ iff $n^2 = 0 \bmod 4$; and $n = 1 \bmod 2$ iff $n^2 = 1 \bmod 4$. As an immediate application of the theorem one has the following.

Theorem 2. *$\mathbb{Q}^3$ has an additive 2-coloring.*

Proof. If $\xi = (a/d,\ b/d,\ c/d) \in \mathbb{Q}^3$ with $\|\xi\| = 1$, then it may be assumed that the integers satisfy $(a, b, c, d) = 1$ and that

$$a^2 + b^2 + c^2 = d^2.$$

Considering the equation $a^2 + b^2 + c^2 = d^2 \bmod 4$, it can be assumed that $d = 1 \bmod 2$ since $(a, b, c, d) = 1$. Hence, $\Sigma_0(\mathbb{Q}^3) \subset \mathbb{Z}^3_{(2)}$. Furthermore, it follows that exactly one of the integers a, b, or c must equal $1 \bmod 2$ and hence $[a + b + c] = [1]$.

Let $s: \mathbb{Q}^3 \to \mathbb{Q}$, where $s(a, b, c) = a + b + c$, be the additive linear functional which sums the coordinates of $\mathbb{Q}^3$. Since $\Sigma_0(\mathbb{Q}^3) \subset \mathbb{Z}^3_{(2)}$ by the above, the restriction of s together with the map ϵ gives a mapping of groups

$$v = \epsilon \cdot s : \Sigma_0(\mathbb{Q}^3) \to \mathbb{Z}/(2),$$

for which if $\|\xi\| = 1$, then $v(\xi) = [a + b + c] = [1]$. □

Although $\mathbb{Q}^3$ may be 2-colored it does allow equilateral triangles. (But not with sides of rational length, of course, since a dilation would contradict the above theorem.) For example, the points

$$(1, 0, 0), (0, 1, 0) \quad \text{and} \quad (0, 0, 1)$$

give an equilateral triangle for which the length of each side is the irrational number $\sqrt{2}$. That this, in particular, is not the case for a 2-colorable plane F^2 is a consequence of the following theorem.

Theorem 3. *Let F be a field contained in $\mathbb{R}$ and let $r \neq 0$ be a fixed real number that may be realized as the distance between two points of F^2. Let $P_0, \ldots, P_n$ with $P_0 = P_n$ be a sequence of points in the plane F^2 so that $\|P_j - P_{j-1}\| = r$ for each $j = 1, \ldots, n$. Then F^2 is 2-colorable iff $n = 0 \bmod 2$ for any such sequence of points.*

Proof. Assume that there exists $\eta_j \in F(i)$ for $j = 1, \ldots, n$ with $\|\eta_j\| = r$ and

$$0 = \eta_1 + \cdots + \eta_n.$$

Since η_1^{-1} is also in $F(i)$, it follows that

$$0 = \eta_1^{-1} \cdot \eta_1 + \cdots + \eta_1^{-1} \cdot \eta_n \in \Sigma_0(F^2),$$

since $\|\eta_1^{-1} \cdot \eta_j\| = 1/r \cdot r = 1$ for each j. If F is 2-colorable, applying the map v from Theorem 1 shows that $n = 0 \bmod 2$.

Conversely, to say that $r \neq 0$ is realized as the distance between two points of F^2 says there exist some nonzero $\eta \in F^2$ for which $\|\eta\| = r$. If $\sum_{j=1}^n \xi_j = 0$ where $\|\xi_j\| = 1$, then $\sum_{j=1}^n \eta \cdot \xi_j = 0$ yields a sequence of points $P_0, \ldots, P_n$ in F^2 with $P_0 = P_n$ and $\|P_j - P_{j-1}\| = r$ for each j. The assumption on such a sequence assures that $n = 0 \bmod 2$. As in Theorem 1, it follows that for any $\sum_{j=1}^n \xi_j \in \Sigma_0(F^2)$, the assignment $v(\sum_{j=1}^n \xi_j) = [n]$ is well defined and provides an additive 2-coloring. Therefore, F^2 is 2-colorable. □

Note that the theorem demonstrates the fact that no suitable multiplication of vectors in $\mathbb{Q}^3$ can be found to be compatible with the Euclidean norm. For if this were so, then since $\mathbb{Q}^3$ is 2-colorable, the same proof would contradict the existence of equilateral triangles in $\mathbb{Q}^3$.

The next theorem shows that regular n-gons for $n \neq 4$ cannot exist in certain 2-colorable planes.

Theorem 4. *Let F be a field contained in $\mathbb{R}$ for which the plane F^2 is 2-colorable and assume that $\sqrt{2} \notin F$. Then the only regular polygon in the plane F^2 is the square.*

Proof. To say that a regular n-polygon exists in the plane F^2 means we can find a primitive nth root of unity $\xi \in F(i)$ with $n \geq 3$. Hence $\xi^n = 1$ and

$$0 = 1 + \xi + \cdots + \xi^{n-1}.$$

If p is a prime divisor of n for which $n = p \cdot m$ for some positive integer m, then

$$\begin{aligned} 0 = {} & (1 + \xi + \cdots + \xi^{m-1}) + (1 + \xi + \cdots + \xi^{m-1}) \cdot \xi^m + \cdots \\ & + (1 + \xi + \cdots + \xi^{m-1}) \cdot (\xi^m)^{p-1}. \end{aligned}$$

Since $(1 + \xi + \cdots + \xi^{m-1}) \neq 0$, it follows that

$$1 + \xi^m + \cdots + (\xi^m)^{p-1} = 0.$$

Since $\|\xi^j\| = 1$ for all integers j and since F^2 is 2-colorable, it follows that p must be even and hence $n = 2^s$ for some positive integer s. But if $s \geq 3$, then the complex number $\eta = \xi^{2^{s-3}}$ is a primitive 8th root of unity. Hence, $\eta = (1/\sqrt{2}) \times (\pm 1 + \pm \cdot i) \in F(i)$. But this cannot be since $\sqrt{2} \notin F$. Hence $s = 2$ and therefore ξ must be a 4th root of unity. This root corresponds to the square which certainly exists in F^2. $\square$

The theorem gives as a consequence a very geometrical proof of the following classical result [6, p. 4].

Corollary. *The only regular polygon in the rational plane is the square.*

Proof. Since $\mathbb{Q}^2$ is 2-colorable by Theorem 2 and $\sqrt{2} \notin \mathbb{Q}$, the result follows from the theorem. $\square$

The explicit description of the points P in $\Sigma_0(F^d)$ is of interest in its own right since these are the ones that may be reached by a path in F^d from the origin. Of course $\Sigma_0(\mathbb{Q}^1) = \mathbb{Z}$ and using Lagrange's Four Square Theorem, it is easily seen in [3] or [2] that $\Sigma_0(\mathbb{Q}^d) = \mathbb{Q}^d$ for $d \geq 5$. In fact, Chilakamarri in [3] computes the maximum number of points in $\mathbb{Q}^d$ that are pairwise a unit distance apart. This is the clique number of the graph whose vertices are points in $\mathbb{Q}^d$ with vertices adjacent if their distance is 1.

Let $S \subset \mathbb{Z}$ be the multiplicative set given by $S = \{p_1^{\alpha_1} \cdots p_t^{\alpha_t} : \alpha_j$ nonnegative integers and p_j are primes for which $p_j = 1 \bmod 4\ \forall j\}$ and let $\mathbb{Z}_S = \{m/s : m \in \mathbb{Z}, s \in S\}$ denote the localization of $\mathbb{Z}$ at S. Recorded below is the description of $\Sigma_0(\mathbb{Q}^d)$ for $d = 2$, 3, and 4. It will be seen from these that $\mathbb{Q}^4$ has an additive 4 coloring.

Example 1. $\Sigma_0(\mathbb{Q}^2) = \mathbb{Z}_S + \mathbb{Z}_S \cdot i$ [2].

If $\xi = a/c + i \cdot b/c \in \Sigma_0(\mathbb{Q}^2)$ with $\|\xi\| = 1$, then it may be assumed that a, b and c form a primitive Pythagorean triplet. That is,

$$a^2 + b^2 = c^2 \quad \text{and } (a, b, c) = 1.$$

Since $(a, b, c) = 1$, the last equation taken modulo p shows that no prime $p = 3 \bmod 4$ can occur as a factor of the denominator c of such a triplet. Neither can the prime 2. Hence, $\Sigma_0(\mathbb{Q}^2) \subset \mathbb{Z}_S + \mathbb{Z}_S \cdot i$.

To show the opposite inclusion, it suffices to show that $\mathbb{Z}_S \subset \Sigma_0(\mathbb{Q}^2)$. Since $\Sigma_0(\mathbb{Q}^2)$ is a ring, and since $\mathbb{Z} \subset \Sigma_0(\mathbb{Q}^2)$, it therefore suffices to show that for a prime $p = 1 \bmod 4$, $1/p \in \mathbb{Z}_S$. For such a p we have integers a and b with $(a, p) = (b, p) = 1$ and $p^2 = a^2 + b^2$. Therefore, $\xi = a/p + b/p \cdot i$ has norm 1. But then if $\bar{\xi}$ is the complex conjugate of ξ, it follows that $\bar{\xi} + \xi = (2 \cdot a)/p \in \Sigma_0(\mathbb{Q}^2)$. Since $(p, 2a) = 1$, we may find integers m, n so that $m \cdot p + n \cdot 2 \cdot a = 1$. It follows that

$$\frac{1}{p} = \frac{m \cdot p + n \cdot 2 \cdot a}{p} = m + \frac{n \cdot 2 \cdot a}{p} \in \sum_0 (\mathbb{Q}^2),$$

since both m and $(n \cdot 2 \cdot a)/p$ are in the ring $\Sigma_0(\mathbb{Q}^2)$.

Example 2. $\Sigma_0(\mathbb{Q}^3) = \mathbb{Z}^3_{(2)}$.

If $(a/d, b/d, c/d)$ is a unit vector where a, b, c and d are integers for which $(a, b, c, d) = 1$, the equation $a^2 + b^2 + c^2 = d^2$ taken modulo 4 shows that $[d] = [1]$. Hence any point in $\Sigma_0(\mathbb{Q}^3)$ must have an odd denominator.

To show that $\Sigma_0(\mathbb{Q}^3) \supset \mathbb{Z}^3_{(2)}$, it suffices to show that $(1/d, 0, 0) \in \Sigma_0(\mathbb{Q}^3)$ where d is odd. But if $d = e \cdot f$ where $(e, f) = 1$, then $1/e + 1/f = (f + e)/d$ where $(d, f + e) = 1$. As in the above example, one shows that if $(a/d, 0, 0) \in \Sigma_0(\mathbb{Q}^3)$ where $(a, d) = 1$, then $(1/d, 0, 0) \in \Sigma_0(\mathbb{Q}^3)$. Hence, it is sufficient to show that $(1/d, 0, 0) \in \Sigma_0(\mathbb{Q}^3)$ where $d = p^k$ is a positive power k of an odd prime p.

By the previous example, if $p = 1 \bmod 4$ then $(1/d, 0) \in \Sigma_0(\mathbb{Q}^2)$ and therefore $(1/d, 0, 0) \in \Sigma_0(\mathbb{Q}^3)$.

For the case that $p = 3 \bmod 4$, a result credited to E. Lionnet [4] states that any odd integer may be written as the sum of four squares two of which are consecutive. Hence d may be written as $d = x^2 + y^2 + z^2 + w^2$ for integers x, y, z, $w \geqslant 0$ and $(x, y, w, z) = 1$. It follows that $d^2 = A^2 + B^2 + C^2$ where $A = (x^2 + y^2 - z^2 - w^2)$, $B = 2 \cdot (x \cdot z + y \cdot w)$, and $C = 2 \cdot (x \cdot w - y \cdot z)$. It may be assumed that $A \geqslant 0$ and since d is odd, $A \neq 0$. If $(d, A) \neq 1$, then p divides $d + A = 2 \cdot (x^2 + y^2)$. Since $p = 3 \bmod 4$, p therefore divides both x and y. But then, since $d = x^2 + y^2 + z^2 + w^2$, p also divides z and w which is impossible. Hence, $(d, A) = 1$.

Since

$$\left(\frac{A}{d}, \frac{B}{d}, \frac{C}{d}\right) + \left(\frac{A}{d}, -\frac{B}{d}, -\frac{C}{d}\right) = \left(\frac{2 \cdot A}{d}, 0, 0\right) \in \Sigma_0(\mathbb{Q}^3),$$

where $(2 \cdot A, d) = 1$, it follows that $(1/d, 0, 0) \in \Sigma_0(\mathbb{Q}^3)$ and therefore completes the assertion.

Example 3. $\Sigma_0(\mathbb{Q}^4)$ is the disjoint union of the sets $A = \mathbb{Z}^4_{(2)}$ and $B = \mathbb{Z}^4_{(2)} + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

Since A is a subgroup in $\mathbb{Q}^4$ and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \notin A$, it follows that the sets are disjoint. Using Example 2, it is clear that all the vectors in the two sets belong to $\Sigma_0(\mathbb{Q}^4)$.

Let a, b, c, d, and e be integers for which $(a, b, c, d, e) = 1$ and $a^2 + b^2 + c^2 + d^2 = e^2$. If e is odd then $1/e \cdot (a, b, c, d) \in A$. If e is even, it follows by considering the equation modulo 8, that $e = 2 \bmod 4$ and that a, b, c, and d are odd. Hence, $e = 2 \cdot f$ where f is odd and therefore $1/e \cdot (a, b, c, d) =$

$$\frac{1}{f} \cdot \left(\frac{a-f}{2}, \frac{b-f}{2}, \frac{c-f}{2}, \frac{d-f}{2}\right) + \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right),$$

where the first term of the sum belongs to A since the differences are all even integers. It is now clear that $\Sigma_0(\mathbb{Q}^4) \subset A \cup B$ which completes the assertion.

Theorem 5. *$\mathbb{Q}^4$ has an additive 4-coloring.*

Proof. From Example 3, it is clear that $\mathbb{Z}^4_{(2)}$ forms a subgroup of $\Sigma_0(\mathbb{Q}^4)$ and that $\Sigma_0(\mathbb{Q}^4) \cong \mathbb{Z}^4_{(2)} \oplus \mathbb{Z}/(2)$. Let $s : \mathbb{Z}^4_{(2)} \to \mathbb{Z}/(2)$ be the map which sums all the coordinates. Then $\epsilon \cdot s : \mathbb{Z}^4_{(2)} \to \mathbb{Z}/(2)$ together with the identity map 1 on $\mathbb{Z}/(2)$, gives a group homomorphism

$$v = (\epsilon \cdot s, 1) : \mathbb{Z}^4_{(2)} \oplus \mathbb{Z}/(2) \to \mathbb{Z}/(2) \oplus \mathbb{Z}/(2).$$

Let $\xi \in \Sigma_0(\mathbb{Q}^4)$ with $\|\xi\| = 1$. According to Example 3, either $\xi = (a/e, b/e, c/e, d/e)$ with e odd or $\xi = (a/e, b/e, c/e, d/e) + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ with e odd. In the former case, since also $a^2 + b^2 + c^2 + d^2 = e^2$, it follows that $a + b + c + d \neq 0 \bmod 2$. Identifying ξ in the direct sum it follows that $v(\xi) \neq 0$.

In the latter case, ξ is identified as $(a/e, b/e, c/e, d/e) \oplus [1]$ and it follows again that $v(\xi) \neq 0$. Hence, v gives the required map. □

It should be noted that since the points $(0, 0, 0, 0)$, $(1, 0, 0, 0)$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ form a tetrahedron in $\mathbb{Q}^4$, the above shows that the chromatic number of $\mathbb{Q}^4$ is 4. This is known ([2], among others). Furthermore, since $\Sigma_0(\mathbb{Q}^d) = \mathbb{Q}^d$ for $d \geq 5$, it follows that $\mathbb{Q}^d$ can have no additive k-coloring for any $k \geq 1$. For if $v : \mathbb{Q}^d \to G$ is an additive k-coloring, then $k \cdot v(1/k, 0, \ldots, 0) = v(1) \neq 0$ and yet $k \cdot G = 0$ gives a contradiction.

The following theorem indicates that an arbitrary k-coloring as well as an additive k-coloring may be extended to a transcendental extension.

Theorem 6. *Let α be a real number transcendental over the field $F \subset \mathbb{R}$. Then F^d can be k-colored (has an additive k-coloring) iff $F(\alpha)^d$ can be k-colored (has an additive k-coloring).*

Proof. If $F(\alpha)^d$ can be k-colored, then surely so can F^d.

To show the converse, since α is transcendental over F, the field $F(\alpha)$ can be viewed as the rational function field $F(x)$ in the indeterminate x. Therefore, let

$$\xi(\alpha) = \left(\frac{p_1(\alpha)}{c(\alpha)}, \ldots, \frac{p_d(\alpha)}{c(\alpha)}\right) \in F(\alpha)^d$$

where $c(x)$ and $p_j(x)$ for $j = 1, \ldots, d$ are polynomials in $F[x]$. If $\|\xi(\alpha)\| = 1$, then it may be assumed that

$$\sum_{j=1}^{d} p_j(x)^2 = c(x)^2$$

and that these polynomials have no common factor.

Setting $x = 0$, one sees that if $c(0) = 0$, then $p_j(0) = 0$ for all j since all the polynomials have real coefficients. But since x would then be a common factor of all the polynomials, it follows that $c(0) \neq 0$. Hence,

$$\xi(0) = \left(\frac{p_1(0)}{c(0)}, \ldots, \frac{p_d(0)}{c(0)}\right) \in F^d \quad \text{and still } \|\xi(0)\| = 1$$

because $\sum_{j=1}^{d} p_j(0)^2 = c(0)^2$.

In order to k-color $F(\alpha)^d$, it is clearly sufficient to k-color $\Sigma_0(F(\alpha)^d)$. Since the evaluation at 0 is linear, the above shows that if $\xi(\alpha) \in \Sigma_0(F(\alpha)^d)$, then $\xi(0) \in \Sigma_0(F^d)$. Assign to $\xi(\alpha)$ the color $\xi(0)$ which it receives from the assumed k-coloring of $\Sigma_0(F^d)$. To show that this gives a k-coloring of $\Sigma_0(F(\alpha)^d)$ it suffices to show that if $\xi_1(\alpha)$, $\xi_2(\alpha) \in \Sigma_0(F(\alpha)^d)$ with $\xi_1(\alpha) - \xi_2(\alpha) = \xi(\alpha)$ and where $\|\xi(\alpha)\| = 1$, then $\xi_1(\alpha)$ and $\xi_2(\alpha)$ are assigned different colors. But then $\xi_1(0)$, $\xi_2(0) \in \Sigma_0(F^d)$ with $\xi_1(0) - \xi_2(0) = \xi(0)$ and where $\|\xi(0)\| = 1$. But since F^d can be k-colored, $\xi_1(0)$ and $\xi_2(0)$ must receive different colors and hence also $\xi_1(\alpha)$ and $\xi_2(\alpha)$.

To show the assertion for an additive k-coloring, it is sufficient to show that the given map $v: \Sigma_0(F^d) \to G$ may be extended to a map $\bar{v}: \Sigma_0(F(\alpha)^d) \to G$. Using the notation above, if $\xi(\alpha) \in \Sigma_0(F(\alpha)^d)$, then define $\bar{v}(\xi(\alpha)) = v(\xi(0))$. Since the evaluation at 0 is a mapping of groups, so is $\bar{v}$. As above, if $\|\xi(\alpha)\| = 1$, then $\|\xi(0)\| = 1$ and it follows that $\bar{v}(\xi(\alpha)) = v(\xi(0)) \neq 0$. □

Remark 1. Given a finitely generated extension $E \subset \mathbb{R}$ over $\mathbb{Q}$, it may be realized as a purely transcendental extension of some finite algebraic extension F over $\mathbb{Q}$. It follows from the above theorem that E^d can be k-colored iff F^d can be k-colored. Given a finite number of points in $\mathbb{R}^d$, one may adjoin the coordinates of these points to form a finitely generated extension E over $\mathbb{Q}$. Using the result of de Bruijn and Erdös as stated in the introduction, it follows that the question of the k-colorability of an arbitrary extension of $\mathbb{Q}$ may be reduced to that of the k-colorability of a finite algebraic extension of $\mathbb{Q}$.

Remark 2. Again using the result of de Bruijn and Erdös and the above theorem, it can be seen that there exist fields F_0 so that F_0^d is maximal in respect to being k-colored provided that $\mathbb{Q}^d$ is k-colorable. Because of the above theorem, $\mathbb{R}$ must also be algebraic over F_0.

3. Real quadratic extensions of $\mathbb{Q}$

Any real quadratic extension of $\mathbb{Q}$ may be realized as $\mathbb{Q}(\sqrt{N})$ where N is a square free positive integer. In this section, the additive k-colorability of the plane $\mathbb{Q}(\sqrt{N})^2$ will be investigated. In what follows, Σ_0 will always denote $\Sigma_0(\mathbb{Q}\sqrt{N})^2$ and for an integer n, $[n]$ will denote the equivalence class of the integer n modulo 2.

The first theorem can be used as a test of whether a plane is 2-colorable.

Theorem 7. *Suppose that the plane F^2 can be 2-colored and hence has a ring homomorphism $v:\Sigma_0(F^2)\to\mathbb{Z}/(2)$. Let $p(x)=a_n\cdot x^n+\cdots+a_0$ be a polynomial so that $a_k\in\Sigma_0(F^2)$ for each $k=1,\ldots,n$.* If *$p(\xi)=0$ for some $\xi\in\Sigma_0(F^2)$ with $v(\xi)=[1]$, then $\sum_{k=0}^{n}v(a_k)=[0]$.*

In particular, if $p(x)=a_n\cdot x^n+\cdots+a_0$ is a polynomial with integral coefficients for which $p(\xi)=0$ for some $\xi\in\mathbb{Q}(i)$ with $\|\xi\|=1$, then $a_n+\cdots+a_0=0 \bmod 2$.

Proof. Since $\xi\in\Sigma_0(F^2)$ so is ξ^k for $k=0,\ldots,n$ and hence $p(\xi)\in\Sigma_0(F^2)$. Clearly, $v(\xi^k)=v(\xi)$ for all k since v is a homomorphism. Applying v to the equation $0=p(\xi)=a_n\cdot\xi^n+\cdots+a_0$ now gives the first result.

If the a_k's are integers then $a_k\in\Sigma_0(F^2)$ and $v(a_k)=[n]$. By assumption $v(\xi)=[1]$ and the second result follows from the first. □

A recent and independent result of Johnson [8] shows in more generality the "if" portion of the following theorem. He shows that the points in $\mathbb{Q}(\sqrt{N})^2$ for $N\neq 3 \bmod 4$ may be partitioned into sets A and B which fail to realize not only the distance 1 but also the distances $\sqrt{p/q}$ where p and q are positive odd integers.

In what follows, recall that for an integer n, $n^2=1 \bmod 8$ iff n is odd, $n^2=4 \bmod 8$ iff $n=2 \bmod 4$, and that $n^2=0 \bmod 8$ iff $n=0 \bmod 4$.

Theorem 8. *Let N be a square free positive integer. Then the plane $\mathbb{Q}(\sqrt{N})^2$ can be 2-colored iff $N\neq 3 \bmod 4$.*

Proof. If N is a positive integer for which $N=3 \bmod 4$, then $N+1=4\cdot a$ for some integer $a>0$. It follows that

$$N=\left[\frac{n+1}{2}\right]^2-\left[\frac{N-1}{2}\right]^2=4\cdot a^2-b^2,$$

where $b=(N-1)/2$ must be odd. But then

$$\xi=\frac{b+\sqrt{N}\cdot i}{2\cdot a}$$

satisfies $\|\xi\|=1$ and is a solution of $a\cdot x^2-b\cdot x+a=0$. Since $2\cdot a-b=1 \bmod 2$, the plane $\mathbb{Q}(\sqrt{N})^2$ where $N=3 \bmod 4$ cannot be 2-colored because otherwise this would contradict Theorem 7.

In order to argue the converse, it suffices by Theorem 1 to find an additive map $v:\Sigma_0\to\mathbb{Z}/(2)$ for which $v(\xi)=[1]$ whenever $\|\xi\|=1$. Let $\mathbb{Z}_{(2)}$, as before, be the subring of those fractions of $\mathbb{Q}$ with odd denominators and ϵ the evaluation $\epsilon:\mathbb{Z}_{(2)}\to\mathbb{Z}/(2)$.

Any $\xi\in\mathbb{Q}(\sqrt{N}, i)$ may be written as

$$\xi=\frac{(a_1+b_1\cdot i)+(a_2+b_2\cdot i)\sqrt{N}}{c},$$

where c, a_j, and b_j for $j=1,2$ are integers and for which it may be assumed that $(a_1,b_1,a_2,b_2,c)=1$. If $\|\xi\|=1$, then $\xi\cdot\bar{\xi}=1$. Performing the multiplication and noticing that $\sqrt{N}\notin\mathbb{Q}$, one obtains the equations

$$a_1^2+b_1^2+N\cdot(a_2^2+b_2^2)=c^2 \quad\text{and} \tag{1}$$

$$a_1\cdot a_2+b_1\cdot b_2=0. \tag{2}$$

The case $N=1 \bmod 4$ will be argued first and the various four exhaustive possibilities for $[a_1]$ and $[b_1]$ considered.

If $[a_1]=[b_1]=[1]$, equation (1) taken modulo 8 gives $c^2=a_1^2+b_1^2+N\cdot(a_2^2+b_2^2)=2+N\cdot(a_2^2+b_2^2) \bmod 8$. Since $N=1 \bmod 4$, it can be seen from this that necessarily $a_2^2=b_2^2=1 \bmod 8$ and that $c^2=4 \bmod 8$. Hence, $[a_1]=[b_1]=[a_2]=[b_2]=[1]$ and $c/2$ is an odd integer. Since $a_1\cdot a_2+b_1\cdot b_2=0$ and since all the integers are assumed to be odd, it follows that

$$\begin{aligned}(a_1+b_1+a_2+b_2)^2&=a_1^2+b_1^2+a_2^2+b_2^2+2\cdot(a_1+a_2)(b_1+b_2)\\&=4 \bmod 8.\end{aligned}$$

Hence, $(a_1+b_1+a_2+b_2)/2$ is also an odd integer and shows

$$\frac{a_1+b_1+a_2+b_2}{c}\in\mathbb{Z}_{(2)} \quad\text{and}\quad \epsilon\left(\frac{a_1+b_1+a_2+b_2}{c}\right)=[1]. \tag{3}$$

But the same conclusion (3) holds true in the remaining possibilities of $[a_1]$ and $[b_1]$. For if $[a_1+b_1]=[1]$, then equation (1) taken modulo 4 shows that it has to be that $[a_2]=[b_2]=[0]$ and that $[c]=[1]$. Hence, c is odd and $[a_1+b_1+a_2+b_2]=[1]$.

Finally, if $[a_1]=[b_1]=[0]$ then Equation (1) gives $c^2=a_2^2+b_2^2 \bmod 4$. If $[c]=0$, then this would mean $[a_2]=[b_2]=[0]$ contradicting the fact that $(a_1, b_1, a_2, b_2, c)=1$. Hence, it must be that $[c]=[1]$ and that $[a_2+b_2]=[1]$. Again, therefore, conclusion (3) holds true.

Using the identification $\mathbb{Q}(\sqrt{N}, i) \cong \mathbb{Q}^4$ along with the linear functional s which sums the coordinates of $\mathbb{Q}^4$, one has an additive map

$$s:\mathbb{Q}(\sqrt{N}, i)\to\mathbb{Q} \quad \text{where } s(\xi)=\frac{a_1+b_1+a_2+b_2}{c}.$$

The results from (3) show that $s(\Sigma_0)\subset\mathbb{Z}_{(2)}$ and that $\epsilon(s(\xi))=[1]$ if $\|\xi\|=1$. Hence,

$$v=\epsilon\cdot s:\sum_0\to\mathbb{Z}/(2)$$

is the desired group map in the case that $N=1 \bmod 4$.

For the case $N=2 \bmod 4$, one notes that if $[a_1]=[b_1]=[1]$, then equation (2) shows that $[a_2]=[b_2]$. It follows by considering equation (1) modulo 4, that no ξ for which $\|\xi\|=1$ and $[a_1]=[b_1]=[1]$ can exist in this case.

If $[a_1+b_1]=[1]$, then equation (1) shows that $[c]=[1]$ and $[a_2]=[b_2]=[0]$. Hence,

$$\frac{a_1+b_1+2\cdot b_2}{c}\in\mathbb{Z}_{(2)} \quad \text{and} \quad \epsilon\left(\frac{a_1+b_1+2\cdot b_2}{c}\right)=[1]. \tag{4}$$

But the same conclusion as in (4) holds for the remaining case of when $[a_1]=[b_1]=[0]$. For then Equation (1) implies that $[c]=[0]$ and that $[a_2]=[b_2]$. It cannot be that $[a_2]=[b_2]=[0]$ because $(a_1, b_1, a_2, b_2, c)=1$. Hence, $[a_2]=[b_2]=[1]$. Observe, therefore, that since $a_1\cdot a_2+b_1\cdot b_2=0$, a_1 and b_1 contain the same power of 2 in their factorization and consequently $a_1+b_1=0 \bmod 4$ or equivalently, $(a_1+b_1)/2$ is an even integer. Furthermore, since also $a_2^2+b_2^2=2 \bmod 8$, equation (1) yields because $N=2 \bmod 4$ that $c^2=0+N\cdot 2=4 \bmod 8$. Hence, $c/2$ is an odd integer. Since $[b_2]=[1]$, the results stated in (4) hold in this case also.

Again identifying $\mathbb{Q}(\sqrt{N}, i)$ with $\mathbb{Q}^4$, one has the map

$$t:\mathbb{Q}(\sqrt{N}, i)\to\mathbb{Q} \quad \text{where}$$

$$t(\xi)=\frac{a_1+b_1+2\cdot b_2}{c}.$$

The results in (4) show therefore, that $t(\Sigma_0)\subset\mathbb{Z}_{(2)}$ and that $\epsilon\cdot t(\xi)=[1]$ if $\|\xi\|=1$. Hence, $v=\epsilon\cdot t:\Sigma_0\to\mathbb{Z}/(2)$ is the required map in the case $N=2 \bmod 4$. □

The equations appearing in the above theorem can be utilized to demonstrate some additional k-colorings of quadratic extensions.

Theorem 9. *Suppose N is a square free positive integer for which $N \neq 2 \bmod 3$. Then the plane $\mathbb{Q}(\sqrt{N})^2$ has an additive 3-coloring.*

It follows that the chromatic number of the plane $\mathbb{Q}(\sqrt{N})^2$ is 3 when $N = 3$ or $7 \bmod 12$.

Proof. As in the above theorem, if

$$\xi = \frac{(a_1 + b_1 \cdot i) + (a_2 + b_2 \cdot i)\sqrt{N}}{c} \in \mathbb{Q}(\sqrt{N}, i)$$

with $\|\xi\| = 1$, then

$$a_1^2 + b_1^2 + N \cdot (a_2^2 + b_2^2) = c^2 \quad \text{and} \tag{1}$$

$$a_1 \cdot a_2 + b_1 \cdot b_2 = 0 \tag{2}$$

where the integers a_1, b_1, a_2, b_2 and c have no common non-trivial divisor.

Let $\mathbb{Z}_{(3)}$ be the local ring of fractions whose denominator is not divisible by 3 and let $\delta : \mathbb{Z}_{(3)} \to \mathbb{Z}/(3)$ be the evaluation at the maximal ideal (3). That is, for $a/b \in \mathbb{Z}_{(3)}$ with $b \neq 0 \bmod 3$, $\delta(a/b) = \bar{a} \cdot (\bar{b})^{-1}$ where $\bar{a}$ and $\bar{b}$ are the equivalence classes of the integers a and b respectively in the ring $\mathbb{Z}/(3)$ of the integers modulo 3.

If $N = 0 \bmod 3$, note that equation (1) when taken modulo 3 shows that $\bar{c} \neq \bar{0}$. For if $\bar{c} = \bar{0}$, then since $\bar{N} = \bar{0}$, $\bar{a}_1^2 + \bar{b}_1^2 = \bar{0}$ which implies that a_1 and b_1 are divisible by 3. It follows that 3^2 divides $N \cdot (a_2^2 + b_2^2)$ and that therefore $\bar{a}_2^2 + \bar{b}_2^2 = \bar{0}$ since N is square free. Hence, a_2 and b_2 are also divisible by 3 which contradicts the assumption that $(a_1, b_1, a_2, b_2, c) = 1$.

Hence, $\bar{c} \neq \bar{0}$ and therefore $\bar{c}^2 = \bar{1}$. It follows again by equation (1) that $\bar{a}_1^2 + \bar{b}_1^2 = \bar{1}$ and that therefore $\bar{a}_1 + \bar{b}_1 \neq \bar{0}$.

Identifying $\mathbb{Q}(\sqrt{N}, i)$ with $\mathbb{Q}^4$ and setting

$$t\left(\frac{a_1, b_1, a_2, b_2}{c}\right) = \frac{a_1 + b_1}{c},$$

the above shows that if $N = 0 \bmod 3$ then $v = \delta \cdot t : \Sigma_0 \to \mathbb{Z}/(3)$ is the desired additive map with $v(\xi) \neq \bar{0}$ whenever $\|\xi\| = 1$.

If $N = 1 \bmod 3$, note that Equation (1) taken modulo 3 shows that

$$\bar{a}_1^2 + \bar{b}_1^2 + \bar{a}_2^2 + \bar{b}_2^2 = \bar{c}^2.$$

If $\bar{c} = \bar{0}$, then at least one of $\bar{a}_1$, $\bar{b}_1$, $\bar{a}_2$, or $\bar{b}_2$ must equal $\bar{0}$. But then Equation (2) set to modulo 3 shows that another must equal $\bar{0}$ also. It follows that a_1, b_1, a_2, b_2 and c are all divisible by 3 contradicting the assumption that these have no nontrivial divisor. Hence, $\bar{c} \neq \bar{0}$ and $\bar{c}^2 = \bar{1}$.

By equation (2) again,

$$(a_1+b_1+a_2+b_2)^2 = a_1^2+b_1^2+a_2^2+b_2^2+2\cdot(a_1+a_2)(b_1+b_2).$$

If $\bar{a}_1+\bar{b}_1+\bar{a}_2+\bar{b}_2=\bar{0}$ then also $(\bar{a}_1+\bar{a}_2)=-(\bar{b}_1+\bar{b}_2)$ and hence $\bar{0}=\bar{1}+(-\bar{2})(\bar{a}_1+\bar{a}_2)^2$. But this is impossible and therefore it must be that $\bar{a}_1+\bar{b}_1+\bar{a}_2+\bar{b}_2\neq\bar{0}$.

It follows in the case $N=1 \bmod 3$ that if

$$s\left(\frac{a_1, b_1, a_2, b_2}{c}\right)=\frac{a_1+b_1+a_2+b_2}{c}, \quad \text{then}$$

$v=\delta\cdot s:\Sigma_0\to\mathbb{Z}/(3)$ is the desired map with $v(\xi)\neq\bar{0}$.

The second part of the theorem now follows from Theorem 8 since to say that $N=3$ or $7 \bmod 12$ is equivalent to asserting that $N=3 \bmod 4$ and $N\neq 2 \bmod 3$. □

The existence of an additive k-coloring on quadratic extensions for those cases not considered by the previous two theorems is covered in the following.

Theorem 10. *Suppose N is a positive square free integer. Then the plane $\mathbb{Q}(\sqrt{N})^2$*

i. *has an additive 4-coloring if $N=3 \bmod 8$.*

ii. *has no additive k-coloring for $1\leq k\leq 6$ if*

$$N=-1 \bmod 24.$$

Proof. Equations (1) and (2) referred to in the following are those as in the previous theorems.

To prove (i), note that if $c=1 \bmod 2$, then equation (1) taken modulo 4 shows that since $N=3 \bmod 4$, either $a_1+b_1=1$ and $a_2=b_2=0 \bmod 2$ or that $a_1=b_1=1 \bmod 2$ and $a_2+b_2=1 \bmod 2$. But the latter situation cannot occur because of equation (2). Hence, if $\epsilon:\mathbb{Z}_{(2)}\to\mathbb{Z}/(4)$ is the map which evaluates at the square of the maximal ideal, one has in this case:

$$\frac{2(a_1+b_1)}{c}\in\mathbb{Z}_{(2)} \quad \text{and } \epsilon\left(\frac{2(a_1+b_1)}{c}\right)\neq 0. \tag{3}$$

If $c=0 \bmod 2$, then not all of the integers a_1, b_1, a_2, and b_2 can equal $0 \bmod 2$. It follows by equation (1) taken modulo 4, that either $a_1+b_1=a_2+b_2=1 \bmod 2$ or that $a_1=b_1=a_2=b_2=1 \bmod 2$. If the latter, then because of equation (1) taken modulo 8 and since $N=3 \bmod 8$, it follows that $c^2=0 \bmod 8$ and therefore $c=4\cdot m$ for some integer m. Since in this case $a_1\neq 0$, one has

$$(a_2, b_2)=\frac{b_2}{a_1}(-b_1, a_1)$$

because of Equation (2). Substituting this into Equation (1) yields

$$(a_1^2+b_2^2N)(a_1^2+b_1^2)=a_1^2\cdot c^2.$$

But since $a_1^2 + b_2^2 N = 4 \bmod 8$ and since $c^2 = 16 \cdot \mathrm{m}^2$, it follows that $a_1^2 + b_1^2 = 0 \bmod 4$ which is impossible since a_1 and b_1 are odd integers.

Hence, it may be assumed that if $c = 0 \bmod 2$, then $a_1 + b_1 = a_2 + b_2 = 1 \bmod 2$. It follows from this and Equation (2) that $a_1^2 + b_1^2 = a_2^2 + b_2^2 = 1$ or $5 \bmod 8$. But then Equation (1) taken modulo 8 shows that since $N = 3 \bmod 8$, $c^2 = 4 \bmod 8$ and hence $c = 2 \cdot m$ where m is an odd integer. It follows again that

$$\frac{2(a_1 + b_1)}{2 \cdot m} = \frac{(a_1 + b_1)}{m} \in \mathbb{Z}_{(2)} \quad \text{and} \quad \epsilon\left(\frac{2(a_1 + b_1)}{c}\right) \neq 0.$$

Identitying $\mathbb{Q}(\sqrt{N}, i)$ with $\mathbb{Q}^4$ and setting

$$t\left(\frac{a_1, b_1, a_2, b_2}{c}\right) = \frac{2 \cdot (a_1 + b_1)}{c},$$

then the above shows that $t : \Sigma_0 \to \mathbb{Z}_{(2)}$ and that $v = \epsilon \cdot t : \Sigma_0 \to \mathbb{Z}/(4)$ is the desired map for which $v(\xi) \neq 0$ if $\|\xi\| = 1$.

To prove (ii), note that since $N = a^2 - b^2$ where $a = (N+1)/2$ and $b = (N-1)/2$, it follows from the assumptions on N that $a = 3 \cdot 4 \cdot m$ for some integer m and that $b = 1 \bmod 2$ and $b = 2 \bmod 3$.

Since for $\xi = (b + \sqrt{N} \cdot i)/a$, $\|\xi\| = 1$, it follows that $\xi + \bar{\xi} = (2 \cdot b)/a = b/(2 \cdot 3 \cdot m) \in \Sigma_0$. But since $(2, b) = 1 = (3, b)$ it must be that $\frac{1}{2}$ and $\frac{1}{3}$ are in Σ_0 as is $\frac{1}{5} \in \Sigma_0(\mathbb{Q}^2) \subset \Sigma_0$. Since Σ_0 is a ring, $1/k \in \Sigma_0$ for $1 \leq k \leq 6$.

As noted before, if v is an additive k-coloring $v : \Sigma_0 \to G$ to some group G of order k, then $k \cdot v(1/k) = v(1) \neq 0$. But $k \cdot G = 0$ and hence no such map can exist. □

Example 4. The integer $167 = 3 \cdot 7 \cdot 2^3 - 1$ is the first one which satisfies the hypothesis of part (ii) of the theorem and also equals $6 \bmod 7$. Therefore, if

$$\xi = \frac{83 + \sqrt{167} \cdot i}{3 \cdot 7 \cdot 2^2}$$

then $\|\xi\| = 1$. Hence, $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{7}$ are in Σ_0.

As in part (ii) of the theorem, this shows that $\mathbb{Q}(\sqrt{167})^2$ has no additive k-coloring for $k < 11$. Hence, even though the plane formed by the quadratic extension $\mathbb{Q}(\sqrt{167})$ can be 7-colored since $\mathbb{R}^2$ can, it has no additive 7-coloring.

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